EXISTENCE AND SMOOTHNESS OF SOLUTIONS TO THE 3D DRIVING-FORCE FREE NAVIER-STOKES EQUATION

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Abstract. Existence of a solution to the driving-force free Navier-Stokes equation with a given initial fluid velocity profile is proven assuming a scalar pressure function and incompressible flow. It is assumed that the fluid is flowing in free space under the forces of viscosity and scalar pressure gradients only, and that there are no external driving forces. Also, it is assumed that the absolute value of the initial velocity profile and all of its spatial derivatives approach zero as $1/(|\mathbf{x}| + a)^\kappa$ as $|\mathbf{x}| \to \infty$, where κ is a constant such that $3/2 < \kappa \leq 2$, and a is a positive constant.

First, we show that for any velocity profile with this spatial characteristic, there exists a scalar pressure gradient with an absolute value that also approaches zero as $1/(|\mathbf{x}| + a)^\kappa$ as $|\mathbf{x}| \to \infty$. We then show that any fluid velocity solution would retain this spatial profile characteristic when propagated in time over a finite interval $0 \leq t \leq T$, provided that solution exists and remains finite. Next, we show that such a solution is bounded over all $\mathbf{x} \in \mathbb{R}^3$ and finite for all finite $t \geq 0$, thereby establishing existence and smoothness. This is done by showing that the global maximum of $(\mathbf{u} \cdot \mathbf{u})$ can grown no faster than time integral of ∇p which is shown to be finite for $t < \infty$. Finally, we show that the solution $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ is unique.

Introduction

The Navier-Stokes equation is one of several equations which governs fluid motion. Essentially, it is a statement of Newton's Second Law ($\mathbf{F} = m\mathbf{a}$) applied to the infinitesimal fluid elements, taking into account the pressure gradients and forces due to viscosity. Proving existence and uniqueness of solutions to this equation with various initial conditions and driving forces has been of great interest to the mathematics community (Ref. 1, 2).

In studying the Navier-Stokes equation, many mathematicians have, over the years, developed the concept of "weak" solutions to help gain insight into the behavior of the equation without the need of finding more "exact" solutions, which may not be possible (Ref. 3). These weak solutions are obtained by relaxing some requirements of the original equation such that solutions are more tractable and easily described. In some cases, it may be possible to demonstrate the existence of a "strong" (or smooth) solution by successive refinements of the weak solutions, or even show that the weak solutions themselves are actually smooth.

In 1934, the French mathematician Jean Leray defined an important class of weak solutions to the Navier-Stokes equation. Instead of working with exact vectors at each point $\mathbf{x} \in \mathbb{R}^3$, the Leray solutions use vector averages over small neighborhoods. Leray showed in his paper that such solutions always exist and never blowup. This achievement opened a new approach to the Navier-Stokes problem. Start with Leray solutions, which you know always exist, and see if you can use them to obtain smooth solutions, which you want to prove always exist.

This and similar methods seem to be the general approach in recent decades to resolving issues about the Navier-Stokes equation and its solutions, including the Millennium Problem sponsored by the Clay Mathematics Institute. Papers implementing these methods, however, are generally extremely difficult to follow, and may be completely understandable only to the authors themselves. This, of course, seriously impedes a proper review of their works since so few are capable of doing so. Also, potential issues in using a particular class of weak solutions may not arise until well after proposed proofs based on these weak solutions have been posted or even published. For example, in October 2018, Tristan Buckmaster and Vlad Vicol of Princeton University showed that under some circumstances, uniqueness of the Leray solutions may break down for the Navier-Stokes equation (Ref. 3, 4). Also, Terance Tao of UCLA constructed a smooth solution to the averaged Navier-Stokes equation that blows up in finite time (Ref. 5).

In this paper, we prove existence and smoothness of solutions to the zero driving-force Navier-Stokes equation for incompressible fluid flow, given a smooth initial fluid velocity profile. We do not, however, follow the general approach of establishing weak solutions, and then somehow showing how to obtain actual or smooth solutions from them (Ref. 6-10). Instead, the proof we present requires only an undergraduate background in calculus, differential equations (ordinary and partial), potential theory, and vector analysis for a reader to follow it.

Problem Description and Main Theorem

Written in vector form, the Navier-Stokes equation is given by

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = \sigma\nabla^2\mathbf{u} - \nabla P + \mathbf{F}(\mathbf{x},t) \tag{1}$$

where $\mathbf{u}$ is the fluid velocity, ρ is the fluid density, P is pressure, σ is the viscosity coefficient, and $\mathbf{F}$ is the external force per unit volume acting on the fluid elements. In addition to satisfying equation (1), a solution $\mathbf{u}$ must also satisfy the equation of continuity, or mass balance, which is given by

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{2}$$

This equation states that whatever net fluid mass (per unit time) flows into a fluid element must appear as increased mass of the element, or equivalently, the mass density at that point in the fluid space.

In the problem we are considering, we assume an incompressible fluid, and therefore the density is constant. In this case, we can write equation (1) as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\nabla^2\mathbf{u} - \nabla p + \mathbf{f}(\mathbf{x},t) \tag{3}$$

where $\nu = \sigma/\rho$ is the normalized viscosity coefficient, $p = (P - P_A)/\rho$ is the normalized pressure, P_A is the ambient pressure (ie. the pressure at infinity), and $\mathbf{f} = \mathbf{F}/\rho$ is the force per unit mass acting on the fluid elements. Also we assume that all external forces acting on the fluid are zero for $t > 0$. That is, we assume that external forces may have acted on the fluid at times $t < 0$, thereby giving rise to an initial fluid velocity profile $\mathbf{u}^0(\mathbf{x})$ at $t = 0$ which we will assume is known. Therefore, equation (3) becomes

$$\frac{\partial \mathbf{u}}{\partial t} \;=\; \nu \nabla^2 \mathbf{u} \;-\; (\mathbf{u}\cdot\nabla)\mathbf{u} \;-\; \nabla p \tag{4a}$$

or equivalently

$$\frac{\partial u_i}{\partial t} \;=\; \nu \nabla^2 u_i \;-\; \sum_{k=1}^{3} u_k \frac{\partial u_i}{\partial x_k} \;-\; \frac{\partial p}{\partial x_i} \tag{4b}$$

for our current problem. The initial condition on $\mathbf{u}$ is given by

$$\mathbf{u}(\mathbf{x},0) \;=\; \mathbf{u}^0(\mathbf{x}) \text{ or } u_i(\mathbf{x},0) \;=\; u_i^0(\mathbf{x}),\; i=1,2,3 \tag{5}$$

where $\mathbf{u}^0(\mathbf{x})$ is a specified vector function of the spatial coordinates. Furthermore, we will assume that $\mathbf{u}^0(\mathbf{x}) \in \mathbb{C}^\infty$ (ie. has continuous partial derivatives to all orders with respect to each spatial variable). For a smooth, physically acceptable solution, we must also assume there exist constants a, C_m, and κ such that

$$\left|\partial_{\mathbf{x}}^m u_i^0(\mathbf{x})\right| \;\le\; \max\left|\frac{\partial^{m=m_1+m_2+m_3} u_i^0}{\partial x^{m_1}\,\partial x^{m_2}\,\partial x^{m_3}}\right| \;\le\; \frac{a^\kappa\, C_m}{(|\mathbf{x}|+a)^\kappa} \tag{6}$$

where $m = m_1 + m_2 + m_3$, $\partial_{\mathbf{x}}^m$ denotes any m^{th} order spatial derivative, and κ can be any constant greater than 3/2. This condition ensures that the initial total energy of fluid motion given by

$$E_0 \;=\; \int_{\mathbb{R}^3} \frac{1}{2}\left|\mathbf{u}^0(\mathbf{x},t)\right|^2 d^3\mathbf{x} \tag{7}$$

is finite. To show this, we insert inequality (6) into (7) and obtain

$$\begin{aligned}
E_0 = \int_{\mathbb{R}^3} \frac{1}{2}\left|\mathbf{u}^0(\mathbf{x},t)\right|^2 d^3\mathbf{x} \;&\le\; \frac{1}{2} a^{2\kappa} C_0^2 \sum_{i=1}^{3} \int_{\mathbb{R}^3} \frac{d^3\mathbf{x}}{(|\mathbf{x}|+a)^{2\kappa}} \\
&=\; 2\pi a^{2\kappa} C_0^2 \sum_{i=1}^{3} \int_0^\infty \frac{r^2}{(r+a)^{2\kappa}}\,dr \;=\; 6\pi a^{2\kappa} C_0^2 \int_0^\infty \frac{r^2}{(r+a)^{2\kappa}}\,dr \\
&=\; 6\pi a^3 C_0^2 \left(\frac{1}{2\kappa-3} - \frac{1}{\kappa-1} + \frac{1}{2\kappa-1}\right) \;=\; \frac{6\pi a^3 C_0^2}{(2\kappa-3)(\kappa-1)(2\kappa-1)}
\end{aligned} \tag{8}$$

From this equation we see that κ must be greater than 3/2 for a finite E_0. Also, as will be shown later, the pressure gradient magnitude $|\nabla p|$ approaches zero as $a^2/(|\mathbf{x}|+a)^2$ as $|\mathbf{x}| \to \infty$ for any such value of $\kappa > 3/2$. Inserting this result into the Navier-Stokes equation, we then show that it implies the fluid velocity components u_i will not in general approach zero as $|\mathbf{x}| \to \infty$ any faster than $a^2/(|\mathbf{x}|+a)^2$, even if the initial conditions are consistent with values of $\kappa > 2$. Therefore, the range of values of κ that would be compatible with a solution propagated in time is given by $3/2 < \kappa \le 2$. [1]

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[1] In many claims of having solved the Navier-Stokes Millennium Problem, the authors state that the exponent $\kappa > 0$, consistent with the Official Problem Statement (Ref. 1). After this, they proceed with a highly esoteric analysis that probably few others can follow. Even if the reader is totally uninformed about their theory and methods, however, there is one aspect of their arguments that is quite noticeable. That is, the exponent κ is used only in defining initial conditions, and the actual analysis is completely independent of this exponent. Therefore, if their proof is correct, then it seems it would be possible to use their methods to "prove" existence and smoothness of infinite energy solutions which are not physically possible.

Now let us consider the issue of $\nabla \cdot \mathbf{u}$ and the pressure gradient ∇p. Since ρ is constant, we see from equation (2) that we must have

$$\nabla \cdot \mathbf{u}(\mathbf{x},t) \;=\; \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k}(\mathbf{x},t) \;=\; 0 \tag{9}$$

in order to satisfy the equation of continuity. Therefore $\mathbf{u}^0(\mathbf{x})$ in equation (5) must be a divergence-free vector function. Taking the divergence of both sides of equation (4a), we have

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) \,+\, \nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u}] \;=\; \nu \nabla^2 (\nabla \cdot \mathbf{u}) \,-\, \nabla^2 p \tag{10}$$

Inserting equation (9) into (10), we obtain

$$\nabla^2 p \,=\, -\nabla \cdot [(\mathbf{u} \cdot \nabla)\mathbf{u}] \tag{11}$$

Carrying out the differentiations indicated on the right hand side of equation (11), and using equation (9), we have

$$\nabla^2 p \;=\; -\sum_{j=1}^{3}\sum_{k=1}^{3} \left(\frac{\partial u_j}{\partial x_k}\right)\left(\frac{\partial u_k}{\partial x_j}\right) \;=\; -\,Q(\mathbf{x},t) \tag{12}$$

(See Ref. 2, p. 35, Ref. 11 Ch. 2, Ref. 12) where we have defined

$$Q(\mathbf{x},t) \;=\; \sum_{j=1}^{3}\sum_{k=1}^{3} \left(\frac{\partial u_j}{\partial x_k}(\mathbf{x},t)\frac{\partial u_k}{\partial x_j}(\mathbf{x},t)\right) \tag{13}$$

Equation (12) governs the pressure needed in order to satisfy equation (9). If the partial derivatives of the u_j and u_k on the right-hand side of equation (12) are known functions of the spatial coordinates x, we can solve this equation as a form of Poisson's equation. From potential theory (Ref. 13, 14, 15), the solution is

$$p(\mathbf{x},t) \;=\; \int_{\mathbb{R}^3} G(\mathbf{x},\mathbf{x}')\, Q(\mathbf{x}',t)\, d^3\mathbf{x}' \;=\; -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{Q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}' \tag{14}$$

where

$$G(\mathbf{x},\mathbf{x}') \;=\; -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{15}$$

is the Greens function associated with the Poisson equation and the boundary condition that the solution approach zero as $|\mathbf{x}|$ approaches infinity. Taking the gradient of both sides of equation (14), we have

$$\nabla p(\mathbf{x},t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} Q(\mathbf{x}',t)\,\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3\mathbf{x}' \tag{16}$$

Equations (14)-(16) are used in the next section to establish the existence and spatial profiles of the scalar pressure p and its gradient ∇p, given the spatial profiles of the fluid velocity $\mathbf{u}$.

At this point, we summarize the problem description by stating our main theorem.

If the Cauchy initial value problem for the incompressible Navier-Stokes equation is defined as

$$\frac{\partial \mathbf{u}}{\partial t} \;=\; \nu\nabla^2\mathbf{u} \,-\, (\mathbf{u}\cdot\nabla)\mathbf{u} \,-\, \nabla p \qquad \textit{where} \qquad \nabla\cdot\mathbf{u}(\mathbf{x},t) \;=\; 0 \tag{4a)(9}$$

with the initial conditions given by

$$\mathbf{u}(\mathbf{x},0) \;=\; \mathbf{u}^0(\mathbf{x}) \text{ or } u_i(\mathbf{x},0) \;=\; u_i^0(\mathbf{x}),\;\; i=1,2,3 \tag{5}$$

where $\mathbf{u}^0(\mathbf{x}) \in \mathbb{C}^\infty$ *is a known divergence-free vector field in* $\mathbb{R}^3$*, and is spatially smooth in the sense that there exists positive constants* a*,* C_m*, and* κ *such that*

$$\left|\partial^m_{\mathbf{x}} u^0_i(\mathbf{x})\right| \;\leq\; \frac{a^\kappa\, C_m}{(|\mathbf{x}|+a)^\kappa} \qquad m = 0,1,2,3,... \quad \textit{and} \quad \frac{3}{2} < \kappa \leq 2 \tag{6}$$

then a solution $\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t)$ *to this problem exists which is also smooth in the sense of the above equation. That is, there exists functions* $C_m(t)$ *such that*

$$\left|\partial^m_{\mathbf{x}} u_i(\mathbf{x},t)\right| \;\leq\; \frac{a^\kappa\, C_m(t)}{(|\mathbf{x}|+a)^\kappa} \qquad m = 0,1,2,3,...$$

These functions need not be continuous, but must be finite for all finite values of t*. Furthermore, the solution* $\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t)$ *is unique.*

Proving this theorem is the objective of this work.

Existence and Uniqueness of Solution

Existence and Spatial Dependence of Scalar Pressure Function. Before demonstrating a solution to the Navier-Stokes equation (4) with the given initial condition and incompressibility constraint, we must first verify that the scalar pressure function p does in fact exist and has the proper spatial dependence for fluid velocity fields $\mathbf{u}(\mathbf{x},t)$ that satisfy

$$|u_i(\mathbf{x},t)| \;\leq\; \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa}\, A\left[u_i\right](t) \tag{17}$$

$$\left|\frac{\partial u_i}{\partial x_j}(\mathbf{x},t)\right| \;\leq\; \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa}\, A\left[\frac{\partial u_i}{\partial x_j}\right](t) \tag{18}$$

and in general

$$\left|\frac{\partial^{m_1+m_2+m_3} u_i}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}(\mathbf{x},t)\right| \;\leq\; \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa}\, A\left[\frac{\partial^{m_1+m_2+m_3} u_i}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}\right](t) \tag{19}$$

where the $A[]$ coefficients may vary with time but not the spatial coordinates.[2] Note that the C_m coefficients from inequality (6) can be used as initial values for the $A[](t)$ functions in (17)-(19). In this section, these inequalities are taken as a given, and we show that the scalar pressure function p and its gradient ∇p exists for fluid velocity spatial profiles satisfying these boundary conditions "at infinity". In the following sections, we use the initial conditions along with the results of this section to show that solutions $\mathbf{u}(\mathbf{x},t)$ to the Navier-Stokes equation do in fact satisfy (17)-(19) for all values of t for which $\mathbf{u}(\mathbf{x},t)$ remains defined.

We start by obtaining expressions, based on the Poisson integral, for p and its spatial derivatives. Let us choose three non-negative integers m_1, m_2, and m_3, and different equation (12) m_1 times with respect to x_1, m_2 times with respect to x_2, and m_3 times with respect to x_3. The result is

$$\nabla^2\left(\frac{\partial^{m_1+m_2+m_3}\; p(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\right) = -\,\frac{\partial^{m_1+m_2+m_3}\; Q(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3} \tag{20}$$

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[2]Throughout this section, we use $A[f]$ to denote a proportionality coefficient associated with the function f enclosed in the square brackets, where f has the property of approaching zero as $1/(|\mathbf{x}|+a)^\kappa$ as $|\mathbf{x}| \to \infty$. This coefficient, which may depend on time but not the spacial coordinates, is defined such that $|f(\mathbf{x},t)| \leq a^\kappa A[f]/(|\mathbf{x}|+a)^\kappa$. This notation was chosen in order to avoid large numbers of variable names and/or subscripts and confusion about their meanings.

Then, using the same potential theory that was used in equation (14), we obtain

$$\frac{\partial^{m_1+m_2+m_3}\ p(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3} = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\ \frac{\partial^{m_1+m_2+m_3}\ Q(\mathbf{x}',t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\ d^3\mathbf{x}' \tag{21}$$

We also obtain the spatial derivatives of the ∇p components (ie. $\partial p/\partial x_i$) by differentiating equation (21) with respect to x_i. The result is

$$\frac{\partial h}{\partial x_i} = \frac{\partial^{m_1+m_2+m_3+1}\ p(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3\partial x_i} = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x_i-x_i'}{|\mathbf{x}-\mathbf{x}'|^3}\ \frac{\partial^{m_1+m_2+m_3}\ Q(\mathbf{x}',t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\ d^3\mathbf{x}' \tag{22}$$

where we have defined the function h as

$$h(\mathbf{x},t) = h[m_1,m_2,m_3](\mathbf{x},t) = \frac{\partial^{m_1+m_2+m_3}\ p(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3} \tag{23}$$

We now differentiate equation (13) m_1 times with respect to x_1, m_2 times with respect to x_2, and m_3 times with respect to x_3 to obtain

$$\begin{aligned}\frac{\partial^{m_1+m_2+m_3}Q(\mathbf{x},t)}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}} &= \sum_{j=0}^{3}\sum_{k=0}^{3}\sum_{\alpha=0}^{m_1}\sum_{\beta=0}^{m_2}\sum_{\gamma=0}^{m_3}\binom{m_1}{\alpha}\binom{m_2}{\beta}\binom{m_3}{\gamma}\frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma+1}u_k(\mathbf{x},t)}{\partial {x_1}^{m_1-\alpha}\partial {x_2}^{m_2-\beta}\partial {x_3}^{m_3-\gamma}\partial x_j}\\ &\quad\times\ \frac{\partial^{m_1+m_2+m_3+1}u_j(\mathbf{x},t)}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial x_k}\end{aligned} \tag{24}$$

where we have used the Leibnitz rule for determining higher derivatives of the product of two functions. The quantities in parentheses to the right of the summation signs are binomial coefficients. Since, by hypothesis, each of the derivatives on the right-hand side of equation (24) approaches zero as $1/(|\mathbf{x}|+a)^\kappa$ as $|\mathbf{x}|$ increases, this equation implies

$$\left|\frac{\partial^{m_1+m_2+m_3}Q(\mathbf{x},t)}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}\right| \ \le\ \frac{a^{2\kappa}B(t)}{(|\mathbf{x}|+a)^{2\kappa}} \tag{25}$$

where

$$\begin{aligned}B(t) &= \sum_{j=i}^{3}\sum_{k=1}^{3}\sum_{\alpha=0}^{m_1}\sum_{\beta=0}^{m_2}\sum_{\gamma=0}^{m_3}\binom{m_1}{\alpha}\binom{m_2}{\beta}\binom{m_3}{\gamma}\\ &\quad\times\ A\left[\frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma+1}u_k(\mathbf{x},t)}{\partial {x_1}^{m_1-\alpha}\partial {x_2}^{m_2-\beta}\partial {x_3}^{m_3-\gamma}\partial x_j}\right]\ A\left[\frac{\partial^{m_1+m_2+m_3+1}u_j(\mathbf{x},t)}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial x_k}\right]\end{aligned} \tag{26}$$

and the $A[]$ coefficients are defined in inequality (19). Taking the absolute value of both sides of equation (21) and using the triangle inequality, we have

$$\begin{aligned}\left|\frac{\partial^{m_1+m_2+m_3}\ p(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\right| &\le \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\ \left|\frac{\partial^{m_1+m_2+m_3}\ Q}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}(\mathbf{x}',t)\right|\ d^3\mathbf{x}'\\ &\le \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\ \frac{a^{2\kappa}B(t)}{(|\mathbf{x}'|+a)^{2\kappa}}\ d^3\mathbf{x}'\end{aligned} \tag{27}$$

Expressing the integral on the right-hand side of this inequality in spherical coordinates, we write

$$\left|\frac{\partial^{m_1+m_2+m_3}\ p(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\right| \ \le\ \frac{1}{4\pi}B(t)\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\ \frac{{r'}^2\sin\theta'}{[r^2+{r'}^2-2rr'\cos\theta']^{1/2}}\,d\phi'\,d\theta'\,dr' \tag{28}$$

Performing the integration over ϕ and making the change of variable $v'=\cos\theta'$ gives us

$$\left|\frac{\partial^{m_1+m_2+m_3}\ p(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\right| \ \le\ \frac{1}{2}B(t)\int_0^\infty\int_{-1}^{1}\frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\ \frac{{r'}^2}{\left[r^2+{r'}^2-2rr'v'\right]^{1/2}}\,dv'\,dr' \tag{29}$$

We now carry out the integration over v' to obtain

$$\begin{aligned}
\left|\frac{\partial^{m_1+m_2+m_3}\ p(\mathbf{x},t)}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\right| &\le \frac{1}{2}\,B(t)\int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,\frac{\left[r^2+r'^2-2rr'v'\right]^{1/2}r'^2}{-rr'}\,|_{-1}^{1}\,dr' \\
&= \frac{1}{2}\,B(t)\int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,\frac{r+r'-|r-r'|}{r}\,r'\,dr' \qquad (30)\\
&= \frac{B(t)}{r}\int_0^r \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,r'^2\,dr' \ + \ B(t)\int_r^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,\frac{r'^2}{r'}\,dr'
\end{aligned}$$

Since $r'>r$ in the second term on the right-hand side of this inequality, we have

$$\begin{aligned}
|h| &\le \frac{B(t)}{r}\int_0^r \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,r'^2\,dr' \ + \ \frac{B(t)}{r}\int_r^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,r'^2\,dr' \\
&= \frac{B(t)}{r}\int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,r'^2\,dr' \ = \ \frac{a^3\ B(t)}{(2\kappa-3)(\kappa-1)(2\kappa-1)\,r} \qquad (31)
\end{aligned}$$

From this inequality, we see that p and its spatial derivatives approach zero at least as fast as $1/r$ as r gets larger.

Let us now show that the spatial derivatives of ∇p must approach zero as $1/r^2$ as $r\to\infty$. Differentiating equation (25) with respect to x_i $(i=1,2,3)$, we have

$$\begin{aligned}
\frac{\partial h}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\partial^{m_1+m_2+m_3}\ p}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\right) = \frac{\partial^{m_1+m_2+m_3}}{\partial^{m_1}x_1\partial^{m_2}x_2\partial^{m_3}x_3}\left(\frac{\partial p}{\partial x_i}\right) \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x_i-x_i'}{|\mathbf{x}-\mathbf{x}'|^3}\,\frac{\partial^{m_1+m_2+m_3}\ Q}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}(\mathbf{x}',t)\ d^3\mathbf{x}' \qquad (32)
\end{aligned}$$

Thus far, we have not made any assumptions about the orientation of the coordinate axises. Therefore, let us define our coordinate axises such that the point $\mathbf{x}$ is on the positive x_3 axis. In this case, the radial direction is along $+x_3$, and we may write

$$\mathbf{x} \ = \ r\,\mathbf{e}_3 \ = \ r\,\mathbf{e}_r \text{ or equivalently } x_1 \ = \ 0,\ x_2 \ = \ 0,\ x_3 \ = \ r \qquad (33)\text{-}(35)$$

where $\mathbf{e}_3$ and $\mathbf{e}_r$ are unit vectors in the x_3 and radial directions respectively. For the primed coordinates, we have

$$x_1' = r'\sin\theta'\cos\phi',\ x_2' = r'\sin\theta'\sin\phi',\ x_3' = r'\cos\theta' \qquad (36)\text{-}(38)$$

Inserting equations (33)-(38) into (32) and setting $i=3$, we obtain

$$\begin{aligned}
\frac{\partial h}{\partial x_3}(\mathbf{x},t) \ &= \ -\frac{1}{4\pi}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{\partial^{m_1+m_2+m_3}\ Q}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}(r'\sin\theta'\cos\phi',r'\sin\theta'\sin\phi',r'\cos\theta',t) \\
&\times\ (r-r'\cos\theta')\,r'^2\sin\theta'\left[r^2+r'^2-2rr'\cos\theta'\right]^{3/2}\,d\phi' d\theta' dr' \ = \ \frac{\partial h}{\partial r} \qquad (39)
\end{aligned}$$

where we have used equations (36)-(38) to express the (Cartesian) components of $\mathbf{x}'$ in terms of the primed spherical coordinates. We will later show that this radial component of ∇h is in fact the dominant component in the limit of large values of $|\mathbf{x}|$. Taking the absolute value of both sides of equation (39) and using the triangle inequality gives us

$$\begin{aligned}
\left|\frac{\partial h}{\partial r}\mathbf{x},t)\right| &\le \frac{1}{4\pi}\int_0^\infty\int_0^\pi\int_0^{2\pi}\left|\frac{\partial^{m_1+m_2+m_3}\ Q}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}(r'\sin\theta'\cos\phi',r'\sin\theta'\sin\phi',r'\cos\theta',t)\right| \\
&\times\ \left|\frac{(r-r'\cos\theta')r'^2\sin\theta'}{\left[r^2+r'^2-2rr'\cos\theta'\right]^{3/2}}\right|\ d\phi' d\theta' dr' \qquad (40)
\end{aligned}$$

Inserting inequality (25) into (40), we then have

$$\left|\frac{\partial h}{\partial r}(\mathbf{x},t)\right| \ \le \ \frac{1}{4\pi}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{a^{2\kappa}B(t)}{(r'+a)^{2\kappa}}\ \frac{|r-r'\cos\theta'|\,r'^2\sin\theta'}{\left[r^2+r'^2-2rr'\cos\theta'\right]^{3/2}}\ d\phi' d\theta' dr' \qquad (41)$$

Performing the integration with respect to ϕ' in this inequality, we obtain

$$\left|\frac{\partial h}{\partial r}\mathbf{x},t)\right| \;\leq\; \frac{1}{2}\,B(t)\int_0^\infty\int_0^\pi \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,\frac{|r-r'\cos\theta'|\,{r'}^2\sin\theta'}{\left[r^2+{r'}^2-rr'\cos\theta'\right]^{3/2}}\,d\theta'\,dr' \tag{42}$$

If we define

$$L(r) \;=\; \frac{1}{2}\int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,J(r,r')dr' \tag{43}$$

where

$$J(r,r') \;=\; \int_0^\pi \frac{|r-r'\cos\theta'|\,{r'}^2\sin\theta'}{\left[r^2+{r'}^2-2rr'\cos\theta'\right]^{3/2}}d\theta' \tag{44}$$

Then we may write inequality (42) as

$$\left|\frac{\partial h}{\partial r}(\mathbf{x},t)\right| \leq B(t)L(r) \;=\; \frac{1}{2}\,B(t)\int_0^\infty \frac{a^{2\kappa}}{(r'+a)^{2\kappa}}\,J(r,r')\,dr' \tag{45}$$

Let us now evaluate the integral in this equation. We first consider the case of $r' < r$. In this case equation (44) can be written as

$$\begin{aligned}
J(r,r') &= \int_{-1}^{1} \frac{r'^2(r-r'v)}{\left[r^2+r'^2-2rr'\right]^{3/2}}\,dv = \frac{r'^2}{2r}\int_{-1}^{1}\frac{2r^2-2rr'v}{\left[r^2+r'^2-2rr'\right]^{3/2}}\,dv \qquad (46)\\
&= \frac{r'^2}{2r}\left[\int_{-1}^{1}\frac{r^2+r'^2-2rr'v}{\left[r^2+r'^2-2rr'\right]^{3/2}}\,dv+\int_{-1}^{1}\frac{r^2-r'^2}{\left[r^2+r'^2-2rr'\right]^{3/2}}\,dv\right]\\
&= \frac{r'^2}{2r}\left[\int_{-1}^{1}\left[r^2+r'^2-2rr'v\right]^{-1/2}\,dv+\int_{-1}^{1}\frac{r^2-r'^2}{\left[r^2+r'^2-2rr'\right]^{3/2}}\,dv\right]\\
&= \frac{r'^2}{2r}\left[\left(-\frac{\left[r^2+r'^2-2rr'v\right]^{1/2}}{rr'}\right)|_{-1}^{1}+\left(\frac{r^2-r'^2}{rr'}\left[r^2+r'^2-2rr'\right]^{-1/2}\right)|_{-1}^{1}\right]\\
&= \frac{r'^2}{2r}\left[2\left(\frac{r+r'-(r-r')}{rr'}\right)\right] = 2\frac{r'^2}{r^2}
\end{aligned}$$

where we have made the change of variable $v' = \cos\theta'$. For $r < r'$, the factor $r - r'\cos\theta'$, whose absolute value appears in equations (41)-(44), is less than zero for values of $v' = \cos\theta' > r/r'$. Therefore, we must change the sign of the integrand at $v' = r/r'$ when evaluating $J(r,r')$. This function for $r < r'$ then becomes

$$\begin{aligned}
J(r,r') &= \int_{-1}^{r/r'}\frac{r'^2(r-r'v)}{\left[r^2+r'^2-2rr'\right]^{3/2}}\,dv-\int_{r/r'}^{1}\frac{r'^2(r-r'v)}{\left[r^2+r'^2-2rr'\right]^{3/2}}\,dv \qquad (47)\\
&= \frac{r'^2}{2r}\left[\frac{r'^2}{2r}\left(-\frac{\left[r^2+r'^2-2rr'v\right]^{1/2}}{rr'}\right)|_{-1}^{r/r'}-\left(\frac{r^2-r'^2}{rr'}\left[r^2+r'^2-2rr'\right]^{-1/2}\right)|_{-1}^{r/r'}\right]\\
&+\frac{r'^2}{2r}\left[\frac{r'^2}{2r}\left(\frac{\left[r^2+r'^2-2rr'v\right]^{1/2}}{rr'}\right)|_{r/r'}^{1}+\left(\frac{r^2-r'^2}{rr'}\left[r^2+r'^2-2rr'\right]^{-1/2}\right)|_{r/r'}^{1}\right]\\
&= 2\frac{r'^2}{r^2}\left(1-\sqrt{1-\left(\frac{r}{r'}\right)^2}\right)
\end{aligned}$$

Let us check continuity of this function near $r = 0$ by evaluating

$$\lim_{r\to0}J(r,r') = \lim_{r\to0}2\frac{r'^2}{r^2}\left(1-\sqrt{1-\left(\frac{r}{r'}\right)^2}\right) = \lim_{s\to1}2\frac{1-s}{1-s^2} = \lim_{s\to1}\frac{2}{1+s} = 1 \tag{48}$$

where we have made the change of variable

$$s = \sqrt{1 - \left(\frac{r}{r'}\right)^2}$$

Since $J(r, r')$ has a finite limit as r approaches zero for any value of $r' > r$, this function is continuous and therefore can be integrated with respect to r near $r = 0$. From equations (46) and (47), we see that

$$J(r, r') \le 2\frac{r'^2}{r^2} \tag{49}$$

if either $r < r'$ or $r > r'$. Inserting inequality (49) into equation (43) we obtain

$$L(r) \;\le\; \frac{1}{r^2}\int_0^\infty \frac{a^{2\kappa} r'^2}{(r' + a)^{2\kappa}}\, dr' \;=\; \frac{a^3}{(2\kappa - 3)(\kappa - 1)(2\kappa - 1)\, r^2} \tag{50}$$

which shows the $1/r^2$ asymptotic behavior of $L(r)$ in the limit as $r \to \infty$. At first sight of inequality (50), one might believe that it implies a singularity exists at $r = 0$. This "singularity", however, is merely an artifact of our gross over-estimation of $J(r, r')$ near $r = 0$. As we have already shown, $J(r, r')$ remains continuous and integrable near $r = 0$. Inserting this result into inequality (45) then gives us

$$\left|\frac{\partial h}{\partial r}(\mathbf{x}, t)\right| \le B(t)L(r) \le \frac{a^3\ B(t)}{(2\kappa - 3)(\kappa - 1)(2\kappa - 1)\, r^2} \tag{51}$$

Thus, we see that $|\partial h/\partial r|$ approaches zero as $1/r^2$ as $r \to \infty$, and again the left-hand side of this inequality remains bounded and continuous as $r \to 0$.

From inequality (25), we see that the non-homogeneous term on the right-hand side of equation (21) approaches zero as $1/r^{2\kappa}$ as r increases. According to inequality (31), however, h approaches zero as $1/r$ as $r \to \infty$. Therefore, the non-homogeneous term in equation (21) can in general be made arbitrarily small compared with the function h and its derivatives by choosing r sufficiently large. This implies that h must approach a harmonic function (ie. solution of Laplace's equation $\nabla^2 h = 0$) in the limit as $r \to \infty$. Let h_L be the (harmonic) function that describes the asymptotic behavior of h as $r \to \infty$. That is h_L is the function to which h approaches as r increases. Since h_L is a harmonic function that approaches zero as $r \to \infty$, it can be written as

$$h_L(r, \theta, \phi) \;=\; \sum_{l=0}^{\infty}\sum_{m=-l}^{l} D_{lm} r^{-(l+1)} Y_{lm}(\theta, \phi) \tag{52}$$

where the D_{lm} are constants and the Y_{lm} are the spherical harmonics. Taking the gradient of both sides of this equation, we have

$$\nabla h_L \;=\; \sum_{l=0}^{\infty}\sum_{m=-l}^{l} D_{lm} r^{-(l+2)} \left[-(l+1) Y_{lm}(\theta, \phi)\, \mathbf{e}_r \;+\; \frac{\partial Y_{lm}}{\partial \theta}(\theta, \phi)\, \mathbf{e}_\theta \;+\; \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \phi}(\theta, \phi)\, \mathbf{e}_\phi \right] \tag{53}$$

Examining equations (52) and (53), we see that the dominate terms (at large values of r) in h_L and ∇h_L are those with $l = m = 0$. Therefore, the asymptotic behavior of h and ∇h can be expressed as

$$h \;\to\; -\frac{D_{00}}{\sqrt{4\pi} r} \text{ and } \nabla h \;\to\; -\frac{D_{00}}{\sqrt{4\pi} r^2}\, \mathbf{e}_r \tag*{(54),(55)}$$

in the limit as $r \to \infty$, with a properly chosen constant D_{00}.[3] Also, note that equations (54) and (55) are consistent with inequalities (31) and (51) respectively for large values of r.

From equation (55), we see that in the limit as $r = |\mathbf{x}| \to \infty$, ∇h approaches a vector function with only a radial component. This implies that there must be a value r_1 such that for $r > r_1$, we have

$$|\nabla h \cdot \mathbf{e}_\theta| \;<\; |\nabla h \cdot \mathbf{e}_r|, \text{ and } |\nabla h \cdot \mathbf{e}_\phi| \;<\; |\nabla h \cdot \mathbf{e}_r| \qquad (56),(57)$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors in the polar and azimuthal directions respectively. Therefore $\nabla h \cdot \mathbf{e}_r$, $\nabla h \cdot \mathbf{e}_\theta$, and $\nabla h \cdot \mathbf{e}_\phi$ are the components of ∇h in the radial, polar, and azimuthal directions respectively. The absolute value of ∇h is given by

$$|\nabla h| \;=\; \sqrt{(\nabla h \cdot \mathbf{e}_r)^2 \,+\, (\nabla h \cdot \mathbf{e}_\theta)^2 \,+\, (\nabla h \cdot \mathbf{e}_\phi)^2} \qquad (58)$$

Inserting (56) and (57) into (58), we have

$$\begin{aligned} |\nabla h| \;&=\; \sqrt{(\nabla h \cdot \mathbf{e}_r)^2 \,+\, (\nabla h \cdot \mathbf{e}_r)^2 \,+\, (\nabla h \cdot \mathbf{e}_r)^2} \;\le\; \sqrt{3\,(\nabla h \cdot \mathbf{e}_r)^2} \qquad (59)\\ &=\; \sqrt{3}\,|\nabla h \cdot \mathbf{e}_r| \;=\; \sqrt{3}\left|\frac{\partial h}{\partial r}\right| \end{aligned}$$

for $r > r_1$. Let us define $r_0 = \max[r_1, a]$. We then have from inequality (50)

$$L(r) \;\le\; \frac{a^3}{3r^2} \;=\; \frac{4a^3}{3(2r)^2} \;\le\; \frac{4a^3}{3(r+r_0)^2} \;\le\; \frac{4a^3}{3(r+a)^2} \text{ if } r > r_0 \qquad (60)$$

If $r < r_0$, we define $L_{\max}$ as the maximum of L over the radial interval $0 \le r \le r_0$. Then we may write

$$L(r)(r+r_0)^2 \;\le\; 4L_{\max} r_0^2 \text{ which implies } L(r) \;\le\; \frac{4L_{\max} r_0^2}{(r+r_0)^2} \;\le\; \frac{4L_{\max} r_0^2}{(r+a)^2} \qquad (61)$$

for $r < r_0$. Combining our results from inequalities (60) and (61), we have

$$L(r) \;\le\; \frac{a^2}{(r+a)^2} A[L] \qquad (62)$$

where we have defined

$$A[L](m_1, m_2, m_3) \;=\; 4 \max\left[L_{\max}\frac{r_0^2}{a^2}, \frac{a}{3}\right] \qquad (63)$$

and m_1, m_2, and m_3 are the positive integers chosen for equation (20). Multiplying both sides of inequality (51) by $\sqrt{3}$ and using (59) then gives us

$$|\nabla h| \;\le\; \frac{\sqrt{3}a^2}{(r+a)^2}\, A[L](m_1, m_2, m_3)\, B(t) \;=\; \frac{\sqrt{3}a^2}{(|\mathbf{x}|+a)^2}\, A[L](m_1, m_2, m_3)\, B(t) \qquad (64)$$

Since $|\partial h/\partial x_i| \le |\nabla h|$ for $i = 1, 2, 3$, this inequality along with equation (20) imply that

$$\begin{aligned} \left|\frac{\partial^{m_1+m_2+m_3+1}\, p}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}\partial x_i}(\mathbf{x}, t_n)\right| \;&\le\; \frac{a^2}{(|\mathbf{x}|+a)^2}\, A\left[\frac{\partial^{m_1+m_2+m_3+1}\, p}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}\partial x_i}\right](t) \qquad (65)\\ &\le\; \frac{a^\kappa}{(|\mathbf{x}|+a)^\kappa}\, A\left[\frac{\partial^{m_1+m_2+m_3+1}\, p}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}\partial x_i}\right](t) \end{aligned}$$

where we have defined

$$A\left[\frac{\partial^{m_1+m_2+m_3+1}\, p}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}\partial x_i}(\mathbf{x}, t)\right] \;=\; \sqrt{3}\, A[L](m_1, m_2, m_3)\, B(t) \qquad (66)$$

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[3] This result is analogous to the dominance of the monopole term in the far-field (ie. large values of $|\mathbf{x}|$) in an electrostatics problem (See Ref. 15, Chapter 4). In such a problem, h corresponds to the electrostatic potential, ∇h corresponds to the electric field, and the right-hand side of equation (21) corresponds to the charge density.

and $B(t)$ is given in equation (26). Also, we have used the fact that $\kappa \leq 2$ which is part of the hypothesis of our main theorem. From inequality (65) and equation (26), we see that $|\nabla h| \to 0$ as $1/(|\mathbf{x}| + a)^\kappa$ in the limit as $|\mathbf{x}| \to \infty$ provided the absolute value of the spatial derivatives of u_i (to all orders) do so also. Hence, the components of ∇p and their spatial derivatives to all order satisfy the required boundary conditions.

Spatial Dependence of Solution. Before establishing existence of a solution of the given problem, let us consider the maximum velocity spatial profiles we would expect such a solution to have. We start by defining a grid G on a finite time interval $0 \leq \ t \leq T$ which consists of N time values t_n such that

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T \tag{67}$$

where N is a positive integer and T is the arbitrarily chosen length of the solution interval. Let us now define a finite difference approximation $\mathbf{u}^{(G)}$ of the solution $\mathbf{u}$ to equation (4). First, we initialize $\mathbf{u}^{(G)}(\mathbf{x}, 0)$ to $\mathbf{u}^0(\mathbf{x})$, where $\mathbf{u}^0(\mathbf{x})$ is the initial profile of the solution $\mathbf{u}(\mathbf{x}, t)$ given in equation (5). Therefore we write

$$\mathbf{u}^{(G)}(\mathbf{x}, 0) \ = \ \mathbf{u}^0(\mathbf{x}) \text{ or equivalently } u_i^{(G)}(\mathbf{x}, 0) \ = \ u_i^0(\mathbf{x}) \tag{68}$$

Next, we define the function $\mathbf{u}^{(G)}$ at the chosen time grid values t_n for $n \geq 1$ according to the recursion relation

$$\begin{aligned} u_i^{(G)}(\mathbf{x}, t_{n+1}) \ = \ & \left[\nu \nabla^2 u_i^{(G)}(\mathbf{x}, t_n) \ - \ \sum_{k=1}^{3} u_k^{(G)}(\mathbf{x}, t_n) \frac{\partial u_i^{(G)}}{\partial x_k}(\mathbf{x}, t_n) - \frac{\partial p^{(G)}}{\partial x_i}(\mathbf{x}, t_n) \right] \Delta t_n \\ & + \ u_i^{(G)}(\mathbf{x}, t_n) \end{aligned} \tag{69}$$

where

$$\Delta t_n = t_{n+1} - t_n \tag{70}$$

For values of t between t_n and t_{n+1}, we define the linear interpolation in time

$$\mathbf{u}^{(G)}(\mathbf{x}, t) \ = \ \left[\mathbf{u}^{(G)}(\mathbf{x}, t_{n+1}) \ - \ \mathbf{u}^{(G)}(\mathbf{x}, t_n) \right] \frac{t - t_n}{\Delta t_n} \ + \ \mathbf{u}^{(G)}(\mathbf{x}, t_n) \tag{71}$$

Since equations (67)-(71) define a finite difference approximation to the solution of equation (4), we expect the approximation $\mathbf{u}^{(G)}$ to converge to the solution $\mathbf{u}$ in the limit as all of the Δt_n approach zero, provided that $\mathbf{u}$ remains defined on the interval $0 \leq t \leq T$. Although these equations precisely define $\mathbf{u}^{(G)}$ for any time grid G, we have not yet shown that $\mathbf{u}^{(G)}$ is bounded on the given time interval in the limit as the Δt_n approach zero. To prove existence, we must show that the function $\mathbf{u}$ defined as

$$\mathbf{u}(\mathbf{x}, t) \ = \ \lim_{\Delta t_{\max} \to 0} \mathbf{u}^{(G)}(\mathbf{x}, t) \tag{72}$$

(where $\Delta t_{\max}$ is the largest value of Δt_n) does in fact remain bounded on $0 \leq t \leq T$. In this section, however, we are only considering the spatial dependence of the solution assuming it does exist.

At this point, we show by induction that inequality (19) must be true for all values of t_n since, by hypothesis, it is true initially (ie. for $t_0 = 0$). Assuming this inequality is true for some grid time t_n, we write

$$\left| \frac{\partial^{m_1+m_2+m_3} u_i}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}(\mathbf{x}, t_n) \right| \ \leq \ \frac{a^\kappa}{(|\mathbf{x}| + a)^\kappa} \, A \left[\frac{\partial^{m_1+m_2+m_3} u_i}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}} \right](t_n) \tag{73}$$

Differentiating equation (69) m_1 times with respect to x_1, m_2 times with respect to x_2, and m_3 times with respect to x_3 gives us

$$\frac{\partial^{m_1+m_2+m_3}\ u_i^{(G)}}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}(\mathbf{x},t_{n+1}) \ = \ \left[\nu\sum_{k=1}^{3}\frac{\partial^{m_1+m_2+m_3+2}u_i^{(G)}(\mathbf{x},t_n)}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial {x_k}^2} \ - \ \frac{\partial^{m_1+m_2+m_3+1}p^{(G)}}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}\partial x_i}(\mathbf{x},t_n)\right.$$
$$\left. - \sum_{k=1}^{3}\sum_{\alpha=0}^{m_1}\sum_{\beta=0}^{m_2}\sum_{\gamma=0}^{m_3}\binom{m_1}{\alpha}\binom{m_2}{\beta}\binom{m_3}{\gamma}\frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma}u_k^{(G)}(\mathbf{x},t_n)}{\partial {x_1}^{m_1-\alpha}\ \partial {x_2}^{m_2-\beta}\ \partial {x_3}^{m_3-\gamma}}\ \frac{\partial^{\alpha+\beta+\gamma+1}u_i^{(G)}(\mathbf{x},t_n)}{\partial {x_1}^{\alpha}\ \partial {x_2}^{\beta}\ \partial {x_3}^{\gamma}\ \partial x_k}\right]\Delta t_n$$
$$+ \ \frac{\partial^{m_1+m_2+m_3}\ u_i^{(G)}}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}(\mathbf{x},t_n) \tag{74}$$

where again we have used the Leibnitz rule for finding higher derivatives of product functions. As before, the quantities in parentheses to the right of the summation signs are binomial coefficients. Taking the absolute value of both sides of equation (74) and using the triangle inequality, we have

$$\left|\frac{\partial^{m_1+m_2+m_3}\ u_i^{(G)}}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}(\mathbf{x},t_{n+1})\right| \ \le \ \left[\ \sum_{k=1}^{3}\sum_{\alpha=0}^{m_1}\sum_{\beta=0}^{m_2}\sum_{\gamma=0}^{m_3}\binom{m_1}{\alpha}\binom{m_2}{\beta}\binom{m_3}{\gamma}\right.$$
$$\times\ \left|\frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma}u_k^{(G)}(\mathbf{x},t_n)}{\partial {x_1}^{m_1-\alpha}\ \partial {x_2}^{m_2-\beta}\ \partial {x_3}^{m_3-\gamma}}\right|\left|\frac{\partial^{\alpha+\beta+\gamma+1}u_i^{(G)}(\mathbf{x},t_n)}{\partial {x_1}^{\alpha}\ \partial {x_2}^{\beta}\ \partial {x_3}^{\gamma}\ \partial x_k}\right| \tag{75}$$
$$\left. +\ \nu\sum_{k=1}^{3}\left|\frac{\partial^{m_1+m_2+m_3+2}u_i^{(G)}(\mathbf{x},t_n)}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial x_k^2}\right| \ + \ \sum_{k=1}^{3}\left|\frac{\partial^{m_1+m_2+m_3+1}p^{(G)}(\mathbf{x},t_n)}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial x_i}\right|\ \right]\Delta t_n$$
$$+\ \left|\frac{\partial^{m_1+m_2+m_3}u_i^{(G)}(\mathbf{x},t_n)}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}\right|$$

Inserting inequalities (65) and (73) into (75), we obtain

$$\left|\frac{\partial^{m_1+m_2+m_3}\ u_i^{(G)}}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}(\mathbf{x},t_{n+1})\right| \ \le \ \left(\ \sum_{k=1}^{3}\sum_{\alpha=0}^{m_1}\sum_{\beta=0}^{m_2}\sum_{\gamma=0}^{m_3}\binom{m_1}{\alpha}\binom{m_2}{\beta}\binom{m_3}{\gamma}\right. \tag{76}$$
$$\times\ \frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}}\ A\left[\frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma}\quad u_k}{\partial {x_1}^{m_1-\alpha}\ \partial {x_2}^{m_2-\beta}\ \partial {x_3}^{m_3-\gamma}}\right](t_n)\ A\left[\frac{\partial^{\alpha+\beta+\gamma+1}\quad u_i}{\partial {x_1}^{\alpha}\ \partial {x_2}^{\beta}\ \partial {x_3}^{\gamma}\ \partial x_k}\right](t_n)$$
$$\left. +\ \frac{a^{\kappa}}{(|\mathbf{x}|+a)^{\kappa}}\ \nu\sum_{k=1}^{3}A\left[\frac{\partial^{m_1+m_2+m_3+2}\quad u_i}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial x_k^2}\right](t_n) + A\left[\frac{\partial^{m_1+m_2+m_3+1}\quad p}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial x_i}\right](t_n)\ \right)\Delta t_n$$
$$+\ \frac{a^{\kappa}}{(|\mathbf{x}|+a)^{\kappa}}A\left[\frac{\partial^{m_1+m_2+m_3}\ u_i}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}\right](t_n)$$

Factoring $a^\kappa/(|\mathbf{x}|+a)^\kappa$ from the right-hand side of this inequality, we have

$$\left|\frac{\partial^{m_1+m_2+m_3}\ u_i^{(G)}}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}(\mathbf{x},t_{n+1})\right| \ \le \ \frac{a^{\kappa}}{(|\mathbf{x}|+a)^{\kappa}}\ A\left[\frac{\partial^{m_1+m_2+m_3}\quad u_i}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}\right](t_{n+1}) \tag{77}$$

where we have defined

$$A\left[\frac{\partial^{m_1+m_2+m_3}\quad u_i}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}(t_{n+1})\right] \ = \ \sum_{k=1}^{3}\sum_{\alpha=0}^{m_1}\sum_{\beta=0}^{m_2}\sum_{\gamma=0}^{m_3}\binom{m_1}{\alpha}\binom{m_2}{\beta}\binom{m_3}{\gamma} \tag{78}$$
$$\times A\left[\frac{\partial^{m_1+m_2+m_3-\alpha-\beta-\gamma}\ u_k}{\partial {x_1}^{m_1-\alpha}\ \partial {x_2}^{m_2-\beta}\ \partial {x_3}^{m_3-\gamma}}\right](t_n)\ A\left[\frac{\partial^{\alpha+\beta+\gamma+1}\quad u_i}{\partial {x_1}^{\alpha}\ \partial {x_2}^{\beta}\ \partial {x_3}^{\gamma}\ \partial x_k}\right](t_n)\ \Delta t_n$$
$$+\,\nu\sum_{k=1}^{3}A\left[\frac{\partial^{m_1+m_2+m_3+2}\ u_i}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial x_k^2}\right](t_n)\,\Delta t_n + A\left[\frac{\partial^{m_1+m_2+m_3+1}\quad p}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}\partial x_i}\right](t_n)\,\Delta t_n$$
$$+\,A\left[\frac{\partial^{m_1+m_2+m_3}\ u_i}{\partial {x_1}^{m_1}\partial {x_2}^{m_2}\partial {x_3}^{m_3}}\right](t_n)$$

and have used the fact that

$$\frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}} \;\leq\; \frac{a^{\kappa}}{(|\mathbf{x}|+a)^{\kappa}} \;\leq\; 1$$

Inequality (77), however, is merely inequality (73) with n replaced by $n+1$. Since inequality (6) implies that (73) is true for $n=0$, we have shown inductively that for all n, there exists time-dependent coefficients $A[](t_n)$ such that (73) is true. Since these spatial dependencies of the $u_i^{(G)}$ must hold for any positive integer N, inequality (73) becomes

$$\left|\frac{\partial^{m_1+m_2+m_3}u_i}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}(\mathbf{x},t)\right| \;\leq\; \frac{a^{\kappa}}{(|\mathbf{x}|+a)^{\kappa}}\,A\left[\frac{\partial^{m_1+m_2+m_3}u_i}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}\right](t) \tag{79}$$

in the limit as $\Delta t_{\max}\to 0$, provided the solution indicated in equation (72) actually exists, as previously indicated. Therefore, we have shown that if the solution $\mathbf{u}(\mathbf{x},t)$ exists, then it must be smooth according to inequalities (17)-(19). At this point, we note that suitable $C_m(t)$ functions for the statement of the main theorem can be obtained from the $A[](t)$ functions in inequality (79) by selecting the maximum $A[](t)$ function such that $m=m_1+m_2+m_3$.

In the case where the time-grid solution $\mathbf{u}^{(G)}(\mathbf{x},t)$ does become arbitrarily large as $\Delta t_{\max}\to 0$, the $A[u_i](t)$ (with $m_1=m_2=m_3=0$) from inequality (79) would become infinitely large at some finite value of t. This situation is commonly called a "smooth blowup". We will later show, however, that the solutions $\mathbf{u}(\mathbf{x},t)$ of the current Navier-Stokes problem do in fact remain bounded for all finite t.

Existence of Pressure Gradient Integral over Time. As indicated in the previous section, equations (67)-(71) define a finite difference approximation to the solution $\mathbf{u}$. This solution will exist if we can show that the $\mathbf{u}^{(G)}$ remain bounded in the limit as the time step sizes approach zero. We must first, however, establish that the time integral of the scalar pressure gradient ∇p exists and remains finite over any finite time interval. We start with the original Navier-Stokes equation.

$$\frac{\partial u_i}{\partial t} \;=\; \nu\nabla^2 u_i \;-\; \sum_{k=1}^{3} u_k\frac{\partial u_i}{\partial x_k} \;-\; \frac{\partial p}{\partial x_i} \tag{4b}$$

Multiplying both sides of this equation by u_i and summing over i, we obtain

$$\sum_{i=1}^{3} u_i\frac{\partial u_i}{\partial t} \;=\; \nu\sum_{i=1}^{3} u_i\nabla^2 u_i \;-\; \sum_{i=1}^{3}\sum_{k=1}^{3} u_i u_k\frac{\partial u_i}{\partial x_k} \;-\; \sum_{i=1}^{3} u_i\frac{\partial p}{\partial x_i} \tag{80}$$

Since

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\,u_i^2\right) \;=\; u_i\frac{\partial u_i}{\partial t}$$

Equation (80) can be written as

$$\sum_{i=1}^{3}\frac{\partial}{\partial t}\left(\frac{1}{2}\,u_i^2\right) \;=\; \nu\sum_{i=1}^{3} u_i\nabla^2 u_i \;-\; \sum_{i=1}^{3}\sum_{k=1}^{3} u_k\frac{\partial}{\partial x_k}\left(\frac{1}{2}\,u_i^2\right) \;-\; \sum_{i=1}^{3} u_i\frac{\partial p}{\partial x_i} \tag{81}$$

From elementary vector analysis, we have

$$\nabla\cdot(u_i\nabla u_i) \;=\; u_i\nabla\cdot(\nabla u_i) \;+\; \nabla u_i\cdot\nabla u_i \;=\; u_i\nabla^2 u_i \;+\; \nabla u_i\cdot\nabla u_i$$

and therefore

$$u_i\nabla^2 u_i \;=\; \nabla\cdot(u_i\nabla u_i) \;-\; \nabla u_i\cdot\nabla u_i \;=\; \nabla^2\left(\frac{1}{2}u_i^2\right) \;-\; \nabla u_i\cdot\nabla u_i \tag{82}$$

Inserting this result into equation (81), we obtain

$$\sum_{i=1}^{3} \frac{\partial}{\partial t}\left(\frac{1}{2}\,u_i^2\right) \;=\; \nu \sum_{i=1}^{3} \nabla^2\left(\frac{1}{2}\,u_i^2\right) - \nu \sum_{i=1}^{3} \nabla u_i \cdot \nabla u_i - \sum_{i=1}^{3}\sum_{k=1}^{3} u_k \frac{\partial}{\partial x_k}\left(\frac{1}{2}\,u_i^2\right) - \sum_{i=1}^{3} u_i \frac{\partial p}{\partial x_i} \tag{83}$$

If we define the energy density of fluid motion K as

$$K(\mathbf{x},t) = \frac{1}{2}\sum_{i=1}^{3} (u_i(\mathbf{x},t))^2 = \frac{1}{2}\mathbf{u}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x},t) \tag{84}$$

equation (83) can be written as

$$\frac{\partial K}{\partial t} \;=\; \nu\,\nabla^2 K \;-\; \nu \sum_{i=1}^{3} \nabla u_i \cdot \nabla u_i \;-\; \sum_{i=1}^{3} u_i\,\frac{\partial K}{\partial x_i} \;-\; \sum_{i=1}^{3} u_i\,\frac{\partial p}{\partial x_i}$$

or equivalently

$$\begin{aligned} \frac{\partial K}{\partial t} \;&=\; \nu\nabla^2 K - \nu \sum_{i=1}^{3} \nabla u_i \cdot \nabla u_i - \mathbf{u}\cdot\nabla K - \mathbf{u}\cdot\nabla p \qquad (85)\\ &=\; \nu\nabla\cdot(\nabla K) - \nu \sum_{i=1}^{3} (\nabla u_i \cdot \nabla u_i) - \nabla\cdot[(K+p)\mathbf{u}] \end{aligned}$$

where we have used the fact that $\nabla\cdot\mathbf{u} = 0$ in the last step. Let us now define the total energy of fluid motion as

$$E(t) \;=\; \int_{\mathbb{R}^3} K(\mathbf{x},t)\,d^3\mathbf{x} \tag{86}$$

The initial value E_0 of this function was shown to be finite in equation (7). Let us examine the evolution of the function $E(t)$. Integrating equation (85) over $\mathbb{R}^3$ and using (86) gives us

$$\frac{dE}{dt} \;=\; \nu\int_{\mathbb{R}^3} \nabla\cdot(\nabla K)\,d^3\mathbf{x} \;-\; \nu\sum_{i=1}^{3}\int_{\mathbb{R}^3} (\nabla u_i\cdot\nabla u_i)\,d^3\mathbf{x} \;-\; \int_{\mathbb{R}^3} \nabla\cdot[(p+K)\mathbf{u})]\,d^3\mathbf{x} \tag{87}$$

We now show that the first and third terms on the right-hand side of equation (87) vanish. Integrating equation (86) over the spherical region in $\mathbb{R}^3$ defined by $|\mathbf{x}| \leq R$ we have

$$\begin{aligned} \frac{\partial}{\partial t}\int_{|\mathbf{x}|\leq R} K(\mathbf{x},t)\,d^3\mathbf{x} \;&=\; \nu\int_{|\mathbf{x}|\leq R} \nabla\cdot(\nabla K)\,d^3\mathbf{x} \;-\; \nu\sum_{i=1}^{3}\int_{|\mathbf{x}|\leq R} (\nabla u_i\cdot\nabla u_i)\,d^3\mathbf{x}\\ &\quad -\; \int_{|\mathbf{x}|\leq R} \nabla\cdot[(p+K)\mathbf{u}]\,d^3\mathbf{x} \qquad (88) \end{aligned}$$

Applying the divergence theorem to the first and third terms on the right-hand side of equation (88), we have

$$\begin{aligned} \frac{\partial}{\partial t}\int_{|\mathbf{x}|\leq R} K(\mathbf{x},t)\,d^3\mathbf{x} \;&=\; \nu\int_{|\mathbf{x}|=R} \nabla K\cdot\mathbf{e}_r\,dS \;-\; \nu\sum_{i=1}^{3}\int_{|\mathbf{x}|\leq R} (\nabla u_i\cdot\nabla u_i)\,d^3\mathbf{x}\\ &\quad -\; \int_{|\mathbf{x}|=R} (p+K)\,\mathbf{u}\cdot\mathbf{e}_r\,dS \qquad (89) \end{aligned}$$

where $\mathbf{e}_r$ is the unit vector in the radial direction. Differentiating both sides of equation (84) with respect to x_j gives us

$$\frac{\partial K}{\partial x_j}(\mathbf{x},t) \;=\; \sum_{i=1}^{3} u_i(\mathbf{x},t)\,\frac{\partial u_i}{\partial x_j}(\mathbf{x},t) \tag{90}$$

Since the function $\mathbf{u}$ as defined in equation (72) must be consistent with inequality (79), we take the absolute value of both sides of equation (90) and use inequality (79) along with the triangle inequality to obtain

$$\left|\frac{\partial K}{\partial x_j}(\mathbf{x},t)\right| \;\le\; \sum_{i=1}^{3}|u_i(\mathbf{x},t)|\left|\frac{\partial u_i}{\partial x_j}(\mathbf{x},t)\right| \;\le\; \frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}}\;\sum_{i=1}^{3}A\left[u_i\right](t)\;A\left[\frac{\partial u_i}{\partial x_j}\right](t) \tag{91}$$

From this inequality, we have

$$\begin{aligned}|\nabla K(\mathbf{x},t)| \;&\le\; \sum_{i=1}^{3}\left|\frac{\partial K}{\partial x_i}(\mathbf{x},t)\right| \;\le\; \sum_{i=1}^{3}\sum_{j=1}^{3}|u_i(\mathbf{x},t)|\left|\frac{\partial u_i}{\partial x_j}(\mathbf{x},t)\right| \\ &\le\; \frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}}\;\sum_{i=1}^{3}\sum_{j=1}^{3}A\left[u_i\right](t)\;A\left[\frac{\partial u_i}{\partial x_j}\right](t)\end{aligned} \tag{92}$$

Applying inequality (92) to the first integral on the right-hand side of equation (89) gives us

$$\begin{aligned}\left|\int_{|\mathbf{x}|=R}\nabla K\cdot\mathbf{e}_r\,dS\right| \;&\le\; \sum_{i=1}^{3}\sum_{j=1}^{3}A\left[u_i\right](t)\;A\left[\frac{\partial u_i}{\partial x_j}\right](t)\int_{|\mathbf{x}|=R}\frac{a^{2\kappa}}{(|\mathbf{x}|+a)^{2\kappa}}\,dS \\ &=\; \frac{4\pi R^2 a^{2\kappa}}{(R+a)^{2\kappa}}\sum_{i=1}^{3}\sum_{j=1}^{3}A\left[u_i\right](t)\;A\left[\frac{\partial u_i}{\partial x_j}\right](t)\end{aligned} \tag{93}$$

Taking the limit of both sides of this inequality as $R\to\infty$, we obtain

$$\lim_{R\to\infty}\int_{|\mathbf{x}|=R}\nabla K\cdot\mathbf{e}_r\,dS \;=\; 0 \tag{94}$$

From the last term on the right-hand side of equation (89), we have

$$\begin{aligned}\left|\int_{|\mathbf{x}|=R}(p+K)\,\mathbf{u}\cdot\mathbf{e}_r\,dS\right| \;&\le\; \int_{|\mathbf{x}|=R}|p+K||\mathbf{u}|\,dS \\ &\le\; \int_{|\mathbf{x}|=R}\left(\overline{|p|}(R)+\overline{K}(R)\right)\frac{a^{\kappa}\,A\left[|\mathbf{u}|\right]}{(R+a)^{\kappa}}\,dS \\ &=\; 4\pi R^2\left(\overline{|p|}(R)+\overline{K}(R)\right)\frac{a^{\kappa}\,A\left[|\mathbf{u}|\right]}{(R+a)^{\kappa}} \\ &=\; 4\pi R^2\;\overline{|p|}(R)\;\frac{a^{\kappa}\,A\left[|\mathbf{u}|\right]}{(R+a)^{\kappa}}\;+\;4\pi R^2\;\overline{K}(R)\;\frac{a^{\kappa}\,A\left[|\mathbf{u}|\right]}{(R+a)^{\kappa}}\end{aligned} \tag{95}$$

where we have defined $\overline{|p|}(R)$ and $\overline{K}(R)$ as the average values of $|p|$ and K respectively for $|\mathbf{x}| = R$. Also, we have defined the fluid velocity magnitude coefficient $A[|\mathbf{u}|](t)$ as

$$A[|\mathbf{u}|](t) \;=\; \sqrt{\sum_{k=1}^{3}A^2[u_k](t)} \qquad \text{so that} \qquad |\mathbf{u}(\mathbf{x},t)| \;\le\; \frac{a^{\kappa}}{(|\mathbf{x}|+a)^{\kappa}}\,A\left[|\mathbf{u}|\right] \tag{96}$$

where the $A[u_i](t)$ are from equation (17). In the right-hand side of inequality (95), the scalar pressure p approaches zero as $1/R$ as $R\to\infty$, and the kinetic energy density K approaches zero as $1/R^{2\kappa}$ as $R\to\infty$. Therefore, the first term of the right-hand side of this inequality approaches zero as $1/R^{\kappa-1}$ as $R\to\infty$, and the second term approaches zero as $1/R^{3\kappa-2}$ as $R\to\infty$. Since $\kappa > 3/2$, it follows then that both terms on the right-hand side of inequality (95) vanish as $R\to\infty$. Therefore, we have

$$\lim_{R\to\infty}\int_{|\mathbf{x}|=R}(p+K)\,\mathbf{u}\cdot\mathbf{e}_r\,dS \;=\; 0 \tag{97}$$

Now we take the limit as $R \to \infty$ of both sides of equation (89), and use (94) and (97) to obtain

$$\frac{dE}{dt} \;=\; -\nu \sum_{i=1}^{3} \int_{\mathbb{R}^3} (\nabla u_i \cdot \nabla u_i)\, d^3\mathbf{x} \;=\; -\nu \sum_{i=1}^{3}\sum_{j=1}^{3} \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) \right)^2 d^3\mathbf{x} \tag{98}$$

Integrating equation (98) with respect to time gives us

$$E(t) \;=\; E_0 \;-\; \nu \sum_{i=1}^{3}\sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\; dt' \;\le\; E_0 \tag{99}$$

Since $E(t)$ is bounded below by zero, the summation of the integrals in equation (99) must be finite, and since each these of these integrals is positive, they must all be finite. Therefore, we may write

$$\int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\; dt' \;=\; W_{ij}(t) \tag{100}$$

where each of the $W_{ij}(t)$ functions are finite for all $t > 0$ and as $t \to \infty$.

Let us now establish a connection between the $W_{ij}(t)$ functions and the time integral of the scalar pressure gradient. We first note that since the integrands in equations (99) and (100) are everywhere greater than or equal to zero, we may write

$$\int_0^t \int_{S^3(t')} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \;\le\; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \;=\; W_{ij}(t) \tag{101}$$

where $S^3(t)$ can be any subset of $\mathbb{R}^3$ which may change with time. Let us now show that

$$\int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t')\, \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| d^3\mathbf{x}\; dt' \;\le\; W_{ij}(t) + W_{ji}(t) \quad i,j = 1,2,3 \tag{102}$$

for all $t > 0$. We first define $S^3_{ij}(t)$ as the subset of S^3 (at time t) where $|\partial u_i/\partial x_j|$ is greater than or equal to $|\partial u_j/\partial x_i|$. We may then write

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right| \left| \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| d^3\mathbf{x}\, dt' \;&\le\; \int_0^t \int_{S^3_{ij}(t')} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \\ &+\; \int_0^t \int_{\mathbb{R}^3 - S^3_{ij}(t')} \left(\frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \end{aligned} \tag{103}$$

Since both integrands on the right-hand side of inequality (102) are positive and the subsets $S^3_{ij}(t)$ and $\mathbb{R}^3 - S^3_{ij}(t)$ are both contained within $\mathbb{R}^3$ for any time t, we have

$$\int_0^t \int_{S^3_{ij}(t')} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \;\le\; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \tag{104}$$

and

$$\int_0^t \int_{\mathbb{R}^3 - S^3_{ij}(t')} \left(\frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \;\le\; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \tag{105}$$

Inserting these into inequality (103) then gives us

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t')\, \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| d^3\mathbf{x}\; dt' \;&\le\; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \\ &+\; \int_0^t \int_{\mathbb{R}^3} \left(\frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right)^2 d^3\mathbf{x}\, dt' \end{aligned} \tag{106}$$

From the definition of the $W_{ij}(t)$ functions in equation (100), inequality (106) can be written as

$$\int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| d^3\mathbf{x}\, dt' \;\leq\; W_{ij}(\mathbb{R}^3, t) \,+\, W_{ji}(\mathbb{R}^3, t) \tag{102}$$

thereby proving inequality (102). Applying the triangle inequality to equation (13), we have

$$|Q(\mathbf{x}, t)| \;\leq\; \sum_{i=1}^{3} \sum_{j=1}^{3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t)\; \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t) \right| \tag{107}$$

Integrating this inequality over $\mathbb{R}^3$ and $t > 0$, and using inequality (102) then gives us

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} |Q(\mathbf{x}, t')|\, d^3\mathbf{x}\, dt' \;&\leq\; \sum_{i=1}^{3} \sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}^3} \left| \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t') \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t') \right| d^3\mathbf{x}\, dt' \\ &\leq\; \sum_{i=1}^{3} \sum_{j=1}^{3} [W_{ij}(t) \,+\, W_{ji}(t)] \;=\; 2 \sum_{i=1}^{3} \sum_{j=1}^{3} W_{ij}(t) \end{aligned} \tag{108}$$

Therefore, since each of the $W_{ij}(t)$ on the right-hand side if inequality (108) is finite, the integral of $|Q(\mathbf{x}, t)|$ over any time interval and any subset of $\mathbb{R}^3$ must also be finite. At this point, we show that this result implies that the time integral of $|\nabla p|$ must be finite for all $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$. Applying the triangle inequality to equation (16), we have

$$|\nabla p(\mathbf{x}, t)| \;\leq\; \frac{1}{4\pi} \int_{\mathbb{R}^3} |Q(\mathbf{x}', t)|\, \frac{|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3\mathbf{x}' \;=\; \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|Q(\mathbf{x}', t)|}{|\mathbf{x} - \mathbf{x}'|^2}\, d^3\mathbf{x}' \tag{109}$$

Integrating both sides of this inequality with respect time gives us

$$\int_0^t \left| \nabla p(\mathbf{x}, t') \right|\, dt' \;\leq\; \frac{1}{4\pi} \int_0^t \int_{\mathbb{R}^3} \frac{|Q(\mathbf{x}', t')|}{|\mathbf{x} - \mathbf{x}'|^2}\, d^3\mathbf{x}'\, dt' \;=\; \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_0^t \frac{|Q(\mathbf{x}', t')|}{|\mathbf{x} - \mathbf{x}'|^2}\, dt'\, d^3\mathbf{x}' \tag{110}$$

where we have reversed the order of integration over space and time. This is valid since the solution $\mathbf{u}(\mathbf{x}, t)$ and its spatial derivatives are smooth prior to any blowup. From inequality 110, we then have

$$\int_0^t |\nabla p(\mathbf{x}, t)|\, dt' \;\leq\; \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} \int_0^t |Q(\mathbf{x}', t')|\, dt'\, d^3\mathbf{x}' \;=\; \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2}\, d^3\mathbf{x}' \tag{111}$$

where we have defined

$$q(\mathbf{x}, t) = \int_0^t \left| Q(\mathbf{x}, t') \right|\, dt' \tag{112}$$

To obtain an upper bound on the time integral of $|\nabla p(\mathbf{x}, t)|$, we first choose any finite number R and split the integral in inequality (111) into two integrals as follows

$$\int_{\mathbb{R}^3} \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2}\, d^3\mathbf{x}' \;=\; \int_{|\mathbf{x}-\mathbf{x}'|>R} \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2}\, d^3\mathbf{x}' \;+\; \int_{|\mathbf{x}-\mathbf{x}'|\leq R} \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2}\, d^3\mathbf{x}' \tag{113}$$

From the first integral on the right-hand side of this equation, we have

$$\int_{|\mathbf{x}-\mathbf{x}'|>R} \frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|^2}\, d^3\mathbf{x}' \;<\; \int_{|\mathbf{x}-\mathbf{x}'|>R} \frac{q(\mathbf{x}', t)}{R^2}\, d^3\mathbf{x}' \;<\; \frac{1}{R^2} \int_{\mathbb{R}^3} q(\mathbf{x}', t)\, d^3\mathbf{x}' \tag{114}$$

since $R < |\mathbf{x} - \mathbf{x}'|$ and the integration region described by $\{\mathbf{x}' |\; |\mathbf{x} - \mathbf{x}'| > R\}$ is a subset of $\mathbb{R}^3$. Therefore, according to inequality (114), the first integral on the right-hand side of inequality (113) is finite. Hence, we have

$$\int_{|\mathbf{x}-\mathbf{x}'|>R} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' < \frac{1}{R^2}\int_{\mathbb{R}^3} q(\mathbf{x}',t)\, d^3\mathbf{x}' \;<\; \infty \tag{115}$$

Now let us consider the second integral on the right-hand side of equation (113). We note that this is an improper integral since the integration region contains the $|\mathbf{x}-\mathbf{x}'| = 0$ singularity. Therefore, we must evaluate this integral by excluding from the integration region a small sphere of radius ϵ centered at the singularity (ie. the point $\mathbf{x}$), doing the integral which now excludes the singularity, and then taking the limit as $\epsilon \to 0$. The basic approach we use is to first define a set of concentric spheres $\sigma_0, \sigma_1, \sigma_2, ..., \sigma_N$ with radii $\epsilon = r_0 < r_1 < r_2 < ... < r_N = R$ respectively. We then define a set of N spherical shells $S_1^3, S_2^3, S_3^3, ..., S_N^3$ as the regions between two successive S spheres. That is,

$$\begin{aligned}
S_1^3 &= \text{ Set of all points } \mathbf{x}' \text{ such that } r_0 \;\le\; |\mathbf{x}-\mathbf{x}'| \;\le\; r_1 \\
S_2^3 &= \text{ Set of all points } \mathbf{x}' \text{ such that } r_1 \;\le\; |\mathbf{x}-\mathbf{x}'| \;\le\; r_2 \\
S_3^3 &= \text{ Set of all points } \mathbf{x}' \text{ such that } r_2 \;\le\; |\mathbf{x}-\mathbf{x}'| \;\le\; r_3 \\
&\;\;\vdots \\
S_N^3 &= \text{ Set of all points } \mathbf{x}' \text{ such that } r_{N-1} \;\le\; |\mathbf{x}-\mathbf{x}'| \;\le\; r_N
\end{aligned}$$

Also, we make the following definitions:

$$\begin{aligned}
V_n &= \frac{4}{3}\pi\left(r_n^3 - r_{n-1}^3\right) \;=\; \text{Volume of spherical shell } S_n^3 \;(n = 1,2,3,\dots,N) \\
\Delta r_n &= r_n - r_{n-1} \;=\; \text{Thickness of spherical shell } S_n^3 \;(n = 1,2,3,\dots,N) \\
\overline{q}_n(t) &= \frac{1}{V_n}\int_{S_n^3} q(\mathbf{x}',t)\; d^3\mathbf{x}' \;=\; \text{Mean value of } q(\mathbf{x}',t) \text{ over } S_n^3 \;(n = 1,2,3,\dots,N) \\
\Delta r_{\max} &= \max\,[\Delta r_n \text{ where } n = 1,2,3,\dots,N]
\end{aligned}$$

Note that the $\overline{q}_n(t)$ are all finite since V_n along with the integral of $q(\mathbf{x},t)$ over any subset of $\mathbb{R}^3$ are both finite. With these definitions, the second integral on the right-hand side of equation (113) can be written as

$$\int_{\epsilon\le|\mathbf{x}-\mathbf{x}'|\le R} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \;=\; \sum_{n=1}^{N}\int_{S_n^3} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \tag{116}$$

In each of the integrals on the right-hand side of this equation, the minimum value of $|\mathbf{x}-\mathbf{x}'|$ is r_{n-1}. Therefore, we may write

$$\int_{S_n^3} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \;\le\; \int_{S_n^3} \frac{q(\mathbf{x}',t)}{r_{n-1}^2}\, d^3\mathbf{x}' \;=\; \frac{\overline{q}_n V_n}{r_{n-1}^2} \tag{117}$$

Next, we note that the volume V_n of S_n^3 must be less than the product of the surface area of the outer sphere σ_3 and the thickness Δr_n. That is

$$V_n < 4\pi r_n^2\, \Delta r_n$$

Inserting this inequality into (117), we have

$$\int_{S_n^3} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \;\le\; \frac{\overline{q}_n V_n}{r_{n-1}^2} \;<\; 4\pi\,\overline{q}_n(t)\left(\frac{r_n}{r_{n-1}}\right)^2 \Delta r_n \tag{118}$$

Inserting this result into equation (116) then gives us

$$\int_{\epsilon\le|\mathbf{x}-\mathbf{x}'|\le R} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \;<\; \sum_{n=1}^{N} 4\pi\,\overline{q}_n(t)\left(\frac{r_n}{r_{n-1}}\right)^2 \Delta r_n \tag{119}$$

Note that the sum on the right-hand side of this equation is finite since each of the $\overline{q}_n(t)$ are also finite. Since inequality (119) holds for all sets $\{r_n\}$ such that $\epsilon = r_0 < r_1 < r_2 < ... < r_N = R$, it must also be true in the limit as $\Delta r_{\max} \to 0$ and $N \to \infty$. Therefore, inequality (119) implies that

$$\int_{\epsilon \le |\mathbf{x}-\mathbf{x}'| \le R} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \;\le\; \lim_{\Delta r_{\max}\to 0} \sum_{n=1}^{\infty} 4\pi\, \overline{q}_n(t)\, \Delta r_n \tag{120}$$

where we have used the fact that the ratio r_n/r_{n-1} approaches unity in the limit as $\Delta r_{\max} \to 0$ and $N \to \infty$. Also, the set of $\overline{q}_n(t)$ becomes a continuum function $\overline{q}(r,t)$ and the discrete sum becomes an integral. Therefore, we write

$$\int_{\epsilon \le |\mathbf{x}-\mathbf{x}'| \le R} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \;\le\; 4\pi \int_{\epsilon}^{R} \overline{q}(r,t)\, dr$$

Finally, since $\overline{q}(r,t)$ is finite at $R=0$, the limit as $\epsilon \to 0$ of the right-hand side of this inequality exists and is finite. Therefore, we have

$$\lim_{\epsilon\to 0} \int_{\epsilon \le |\mathbf{x}-\mathbf{x}'| \le R} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x}' \;\le\; 4\pi \int_{0}^{R} \overline{q}(r,t)\, dr \tag{121}$$

which is also finite. Inserting inequalities (113) and (121) into (111) gives us

$$\int_0^t |\nabla p(\mathbf{x},t)|\; dt' \;\le\; \frac{1}{4\pi} \int_{|\mathbf{x}-\mathbf{x}'|>R} \frac{q(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|^2}\, d^3\mathbf{x} \;+\; \int_0^R \overline{q}(r,t)\, dr \tag{122}$$

Since both terms on the right-hand side of this inequality have been shown to be finite, we have shown that the time integral of $|\nabla p(\mathbf{x},t)|$ is finite for all $\mathbf{x}$ and finite t. This is a critical step toward establishing existence and smoothness of $\mathbf{u}(\mathbf{x},t)$ over time.

Existence and Smoothness of Solution over Time. At this point, we show that a solution $\mathbf{u}(\mathbf{x},t)$ which is initially smooth (ie. satisfies the boundary condition that $\mathbf{u}(\mathbf{x},0)$ and its spatial derivatives to all order approach zero as $1/(|\mathbf{x}|+a)^{\kappa}$ as $|\mathbf{x}| \to \infty$) will in fact remain smooth and finite for all $t>0$. From equation (84), we see that $\mathbf{u}(\mathbf{x},t)$ will remain finite if and only if $K(\mathbf{x},t)$ does so also. Therefore, let us show that K is in fact defined over all $\mathbf{x} \in \mathbb{R}^3$ and $t>0$. We first define $\mathbf{x}^*(t)$ as the position of the spatial maximum of K at time t, and $K^*(t)$ as the value of this maximum. If this same spatial maximum occurs at more than one point, then $\mathbf{x}^*(t)$ can be chosen as any one of them. Therefore, we write

$$K^*(t) \;=\; K(\mathbf{x}^*(t),t) \tag{123}$$

These values must, of course, exist initially since $\mathbf{u}(\mathbf{x},0)$ and $K(\mathbf{x},0)$ are smooth by hypothesis. Since K is initially smooth, it will remain so unless a global maximum becomes infinite.[4] Let us determine how $K^*(t)$ evolves in time for a spatially smooth $K(\mathbf{x},t)$. Since, by hypothesis, a maximum of K occurs at $\mathbf{x}^*$ and K is still smooth, we must have

$$\nabla K(\mathbf{x}^*(t),t) \;=\; 0 \quad \text{and} \quad \frac{\partial^2 K}{\partial x_i^2}(\mathbf{x}^*(t),t) \;\le\; 0 \quad (i=1,2,3) \qquad (124),(125)$$

where inequality (125) arises from the second derivative test for spatial maxima. Differentiating equation (123) with respect to t and using equation (124), we obtain

$$\frac{dK^*}{dt} \;=\; \frac{\partial K}{\partial t}(\mathbf{x}^*,t) \;+\; \nabla K(\mathbf{x}^*,t)\cdot\frac{d\mathbf{x}^*}{dt} \;=\; \frac{\partial K}{\partial t}(\mathbf{x}^*,t) \tag{126}$$

[4] Recall that in the section titled *Spatial Dependence of Solution*, it was shown that an initially smooth solution would remain smooth for as long as it's defined.

Also, it should be noted that equation (126) is still valid even if $\mathbf{x}^*$ changes discontinuously in time due to evolution of another maximum point of K at a different location. In this case, the value of $K^*(t_1)$ at the new global maximum location must be the same as $K^*(t_1)$ at the previous location, where we have defined t_1 as the time of the transition. Otherwise, there would be a discontinuity in K and therefore $\mathbf{u}$ at a time when these functions are known to still be smooth.

We next insert $\mathbf{x}^*$ into equation (85) and use equation (126) to obtain

$$\begin{aligned}\frac{dK^*}{dt} \;=\; \frac{\partial K}{\partial t}(\mathbf{x}^*(t),t) \;&=\; \nu\,\nabla^2 K(\mathbf{x}^*(t),t) \;-\; \nu \sum_{i=1}^{3} \nabla u_i(\mathbf{x}^*(t),t)\cdot\nabla u_i(\mathbf{x}^*(t),t) \\ &\quad -\; \mathbf{u}(\mathbf{x}^*(t),t)\cdot\nabla p(\mathbf{x}^*(t),t)\end{aligned} \tag{127}$$

From inequality (125) and the fact that $\nabla u_i \cdot \nabla u_i \geq 0$, equation (127) implies that

$$\frac{dK^*}{dt} \;\leq\; -\,\mathbf{u}(\mathbf{x}^*(t),t)\cdot\nabla p(\mathbf{x}^*(t),t) \tag{128}$$

At this point, we define $K_1^*(t)$ as the maximum value of $K^*(t)$ allowed by inequality (128). Examining this inequality, we see that the largest possible values for the time derivative of K^* occur when $\mathbf{u}(\mathbf{x}^*(t),t)$ and $\nabla p(\mathbf{x}^*(t),t)$ are anti-parallel vectors. In this case, the right-hand side of inequality (128) becomes

$$|\mathbf{u}(\mathbf{x}^*(t),t)||\nabla p(\mathbf{x}^*(t),t)| \;=\; \sqrt{2}\;|\nabla p(\mathbf{x}^*(t),t)|\,\sqrt{K^*(t)}$$

where we have used equation (84). Therefore, the largest possible values at time t for the time derivative of $K^*(t)$ is determined from the ordinary differential equation

$$\frac{dK_1^*}{dt} \;=\; \sqrt{2}\;|\nabla p(\mathbf{x}^*(t),t)|\;\sqrt{K_1^*} \tag{129}$$

Upon integrating this equation with respect to t, we obtain $K_1^*(t)$ which we defined as the greatest possible value of $K^*(t)$. To solve this equation, we divide both sides by $\sqrt{K_1^*}$ and integrate with respect to t to obtain

$$K_1^*(t) \;=\; \frac{1}{2}\left(\int_0^t \left|\nabla p(\mathbf{x}^*(t'),t')\right|\,dt' \;+\; \sqrt{2K_0^*}\right)^2 \tag{130}$$

where K_0^* is the initial value of K^*.

Let us now consider the question of whether K, and therefore $\mathbf{u}$, can reach infinite values at some blowup point $\mathbf{x}_b$ in finite time. If this happens, it must be a "smooth" blowup as indicated in the discussion following inequality (79). Such a blowup would start with a global maximum of K forming at $\mathbf{x}_b$ at some finite time t_b. This time value t_b is not a blowup time (ie. a time when K becomes infinite at $\mathbf{x}_b$), but only a time when the (finite) maximum of K first forms at $\mathbf{x}_b$. For a smooth blowup to occur at this point, the global maximum of K must remain there for an additional "growth time" Δt in order for $K(\mathbf{x}_b,t)$ to reach infinite values. To determine if this happens, we note that from equation (130), we have

$$K_1^*(t_b+\Delta t) \;=\; \frac{1}{2}\left(\int_0^{t_b} \left|\nabla p(\mathbf{x}^*(t'),t')\right|\,dt' \;+\; \int_{t_b}^{t_b+\Delta t} \left|\nabla p(\mathbf{x}_b,t')\right|\,dt' \;+\; \sqrt{2K_0^*}\right)^2 \tag{131}$$

where we have used the fact that $\mathbf{x}_b = \mathbf{x}^*$ if $t > t_b$. The first integral on the right-hand side of this equation must be finite since, by hypothesis, a smooth solution (and therefore a smooth ∇p) exists for all $\mathbf{x}$ prior to time t_b. The second integral must also be finite since

$$\int_{t_b}^{t_b+\Delta t} \left|\nabla p(\mathbf{x}_b,t')\right|\,dt' \;<\; \int_0^{t_b+\Delta t} \left|\nabla p(\mathbf{x}_b,t')\right|\,dt'$$

and according to inequality (122), the integral on the right-hand side of this inequality is finite for all $\Delta t > 0$. Therefore, it is not possible for K_1^* to ever reach infinite values in finite time, which implies that a smooth blowup is not possible.

Hence, we conclude that the solutions for the fluid velocity $\mathbf{u}(\mathbf{x}, t)$ and the scalar pressure function $p(\mathbf{x}, t)$ exist and are smooth for all $t > 0$. Furthermore, equation (98) implies that the total energy of fluid motion E decreases monotonically to zero. Then, since the solution $\mathbf{u}$ has been shown to be smooth, it follows that $\mathbf{u}(\mathbf{x}, t) \to 0$ as $t \to \infty$ for all $\mathbf{x} \in \mathbb{R}^3$. Therefore, the existence of solution part of our main theorem is proven.

Uniqueness of Solution. Let us now show that the solution of the given problem is in fact unique. We start by defining $\mathbf{u}^{(1)}(\mathbf{x}, t)$ and $\mathbf{u}^{(2)}(\mathbf{x}, t)$ along with the corresponding scalar pressure functions $p^{(1)}(\mathbf{x}, t)$ and $p^{(2)}(\mathbf{x}, t)$ as two possible solutions of equation (4) with initial condition (5) and zero-divergence constraint (9). We therefore write

$$\frac{\partial \mathbf{u}^{(1)}}{\partial t} \;=\; \nu \nabla^2 \mathbf{u}^{(1)} - (\mathbf{u}^{(1)} \cdot \nabla)\mathbf{u}^{(1)} - \nabla p^{(1)} \tag{132}$$

and

$$\frac{\partial \mathbf{u}^{(2)}}{\partial t} \;=\; \nu \nabla^2 \mathbf{u}^{(2)} - (\mathbf{u}^{(2)} \cdot \nabla)\mathbf{u}^{(2)} - \nabla p^{(2)} \tag{133}$$

Subtracting equation (132) from (133), we have

$$\frac{\partial \mathbf{D}}{\partial t} = \nu \nabla^2 \mathbf{D} - (\mathbf{u}^{(2)} \cdot \nabla)\mathbf{D} - (\mathbf{D} \cdot \nabla)\mathbf{u}^{(1)} + \nabla p^{(1)} - \nabla p^{(2)} \tag{134}$$

where we have defined

$$\mathbf{D}(\mathbf{x}, t) = \mathbf{u}^{(2)}(\mathbf{x}, t) - \mathbf{u}^{(1)}(\mathbf{x}, t) \tag{135}$$

as the difference between the two solutions. Taking the scalar product of both sides of equation (134) with $\mathbf{D}$, we have

$$\begin{aligned}
\mathbf{D} \cdot \frac{\partial \mathbf{D}}{\partial t} =\; & \nu \mathbf{D} \cdot \nabla^2 \mathbf{D} - \mathbf{D} \cdot \Big[(\mathbf{u}^{(2)} \cdot \nabla)\mathbf{D}\Big] - \mathbf{D} \cdot \Big[(\mathbf{D} \cdot \nabla)\mathbf{u}^{(1)}\Big] + \mathbf{D} \cdot \nabla p^{(1)} - \mathbf{D} \cdot \nabla p^{(2)} \\
= \nu \sum_{i=1}^{3} D_i \nabla^2 D_i & - \sum_{i=1}^{3} \sum_{k=1}^{3} D_i u_k^{(1)} \frac{\partial D_i}{\partial x_k} - \sum_{i=1}^{3} \sum_{k=1}^{3} D_i D_k \frac{\partial u_i^{(1)}}{\partial x_k} - \mathbf{D} \cdot (\nabla p^{(2)} - \nabla p^{(1)}) \\
= \nu \sum_{i=1}^{3} \nabla \cdot (D_i \nabla D_i) & - \nu \sum_{i=1}^{3} (\nabla D_i) \cdot (\nabla D_i) - \mathbf{u}^{(1)} \cdot \nabla \left(\frac{1}{2} \mathbf{D} \cdot \mathbf{D}\right) \\
- \mathbf{D} \cdot \Big[(\mathbf{D} \cdot \nabla)\mathbf{u}^{(1)}\Big] & - \mathbf{D} \cdot (\nabla p^{(2)} - \nabla p^{(1)}) \\
= \nu \sum_{i=1}^{3} \nabla \cdot (D_i \nabla D_i) & - \nu \sum_{i=1}^{3} (\nabla D_i) \cdot (\nabla D_i) + \nabla \cdot \left(\frac{1}{2} \mathbf{D} \cdot \mathbf{D}\right) + \frac{1}{2} (\mathbf{D} \cdot \mathbf{D}) \nabla \cdot \mathbf{u}^{(1)} \\
- \mathbf{D} \cdot \Big[(\mathbf{D} \cdot \nabla)\mathbf{u}^{(1)}\Big] & - \nabla \cdot \Big[(p^{(2)} - p^{(1)})\mathbf{D}\Big] + (p^{(2)} - p^{(1)}) \nabla \cdot \mathbf{D}
\end{aligned} \tag{136}$$

Since $\nabla \cdot \mathbf{u}^{(1)} = 0$ and $\nabla \cdot \mathbf{D} = 0$, the fourth and seventh terms on the right-hand side of this equation vanish, and we write

$$\begin{aligned}
\frac{\partial W_D}{\partial t} = \nu \sum_{i=1}^{3} \nabla \cdot (D_i \nabla D_i) - \nu \sum_{i=1}^{3} (\nabla D_i) \cdot (\nabla D_i) \\
- \nabla \cdot \Big(W_D \mathbf{u}^{(1)}\Big) - \mathbf{D} \cdot \Big[(\mathbf{D} \cdot \nabla)\mathbf{u}^{(1)}\Big] - \nabla \cdot (p_D \mathbf{D})
\end{aligned} \tag{137}$$

where we have defined the normalized energy density W_D associated with $\mathbf{D}$, and pressure difference p_D as

$$W_D = \frac{1}{2}(\mathbf{D}\cdot\mathbf{D}) \qquad \text{and} \qquad p_D = p^{(2)} - p^{(1)} \tag{138),(139}$$

Integrating equation (137) over all $\mathbb{R}^3$ space, we obtain

$$\begin{aligned}\frac{dE_D}{dt} &= \nu\sum_{i=1}^{3}\int_{\mathbb{R}^3}\nabla\cdot(D_i\nabla D_i)\,d^3\mathbf{x} - \nu\int_{\mathbb{R}^3}\sum_{i=1}^{3}(\nabla D_i)\cdot(\nabla D_i)\,d^3\mathbf{x} \\ &- \int_{\mathbb{R}^3}\nabla\cdot\left(W_D\mathbf{u}^{(1)}\right)d^3\mathbf{x} - \int_{\mathbb{R}^3}\mathbf{D}\cdot\left[(\mathbf{D}\cdot\nabla)\mathbf{u}^{(1)}\right]d^3\mathbf{x} - \int_{\mathbb{R}^3}\nabla\cdot(p_D\mathbf{D})\,d^3\mathbf{x}\end{aligned} \tag{140}$$

where we have defined the normalized total energy density associated with $\mathbf{D}$ as

$$E_D(t) = \int_{\mathbb{R}^3} W_D(\mathbf{x},t)\,d^3\mathbf{x} = \frac{1}{2}\int_{\Re^3}\mathbf{D}(\mathbf{x},t)\cdot\mathbf{D}(\mathbf{x},t)\,d^3\mathbf{x} \tag{141}$$

The first, third, and fifth terms on the right-hand side of equation (140) vanish via the divergence theorem and the fact that the integrands in each of these terms approach zero as $1/(|\mathbf{x}|+a)^{2\kappa}$ as $|\mathbf{x}|\to\infty$. Therefore, equation (140) becomes

$$\frac{dE_D}{dt} = -\int_{\mathbb{R}^3}\left(\sum_{i=1}^{3}\nu(\nabla D_i)\cdot(\nabla D_i) + \mathbf{D}\cdot\left[(\mathbf{D}\cdot\nabla)\mathbf{u}^{(1)}\right]\right)d^3\mathbf{x} = Y(t) \tag{142}$$

where we have defined

$$Y(t) = -\int_{\mathbb{R}^3}\left(\sum_{i=1}^{3}\nu(\nabla D_i)\cdot(\nabla D_i) + \mathbf{D}\cdot\left[(\mathbf{D}\cdot\nabla)\mathbf{u}^{(1)}\right]\right)d^3\mathbf{x} \tag{143}$$

Integrating both sides of equation (142) with respect to time, we have

$$E_D(t) = \int_0^t Y(t')\,dt' \tag{144}$$

where we have used the fact that $E_D(0) = 0$ since $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ have the same initial conditions (ie. $\mathbf{u}^{(1)}(\mathbf{x},0) = \mathbf{u}^{(2)}(\mathbf{x},0) = u^0(\mathbf{x})$ at $t=0$ and all $\mathbf{x}\in\mathbb{R}^3$. To determine $E_D(t)$ for $t>0$, let us construct a grid G of discrete time values t'_n on the interval $0\le t'\le t$ such that

$$0 = t'_0 < t'_1 < t'_2 < \;...\; < t'_N = t \tag{145}$$

where N is the number of subintervals defined by G on the interval. We define a finite time difference estimate of the solution of equation (142), or equivalently (144), at the grid times t'_n according to

$$E_D^{(G)}(t'_0) = E_D^{(G)}(0) = E_D(0) = 0 \tag{146}$$

for $n=0$, and

$$E_D^{(G)}(t'_{n+1}) = Y^{(G)}(t'_n)(t'_{n+1}-t'_n) + E_D^{(G)}(t'_n) \tag{147}$$

for $0\le n\le N$. The values $Y^{(G)}(t'_n)$ in this equation are obtained from equation (143), where we set $t = t'_n$ and $\mathbf{D} = \mathbf{D}^{(G)}(\mathbf{x},t'_n)$, where $\mathbf{D}^{(G)}(\mathbf{x},t'_n)$ is the finite difference estimate of $\mathbf{D}$ at time t'_n. Since $\mathbf{D}(\mathbf{x},0)=0$ for all $\mathbf{x}\in\mathbb{R}^3$, equation (143) implies that $Y^{(G)}(0)=0$. Inserting this result into equation (147) with $n=0$, we have $E_D^{(G)}(t'_1)=0$. From equation (141), we then have

$$E_D^{(G)}(t'_1) = \int_{\Re^3} W_D^{(G)}(\mathbf{x},t'_1)\,d^3\mathbf{x} = \frac{1}{2}\int_{\mathbb{R}^3}\mathbf{D}^{(G)}(\mathbf{x},t'_1)\cdot\mathbf{D}^{(G)}(\mathbf{x},t'_1)\,d^3\mathbf{x} = 0 \tag{148}$$

where we have defined $W_D^{(G)}(\mathbf{x},t'_n)$ as the W_D function corresponding to the finite difference approximation at t'_n. Since the integrand in this equation is continuous and greater than or equal to zero at all points $\mathbf{x} \in \mathbb{R}^3$, $E_D^{(G)}(t_1')$ can equal zero only if $\mathbf{D}^{(G)}(\mathbf{x}, t_1') = 0$ at all points $\mathbf{x}$. Inserting this result into equation (143), we then have $Y^{(G)}(t_1') = 0$. This implies (via equation (147)) that $E_D^{(G)}(t_2') = 0$, which in turn implies that $\mathbf{D}^{(G)}(\mathbf{x}, t_2') = 0$ at all points $\mathbf{x} \in \mathbb{R}^3$, and therefore $Y^{(G)}(t_3') = 0$. If we continue in this manner, we may show that

$$Y^{(G)}(t_1') = Y^{(G)}(t_2') = \ \dots \ = Y(t_N') = 0 \tag{149}$$

and

$$E_D^{(G)}(t_1') = E_D^{(G)}(t_2') = \ \dots \ = E_D(t_N') = 0 \tag{150}$$

regardless of the grid time spacing or number of grid points. Therefore, in the limit as maximum difference between successive grid times (ie. max over n of $t_{n+1} - t_n$) approaches zero, these equations become

$$Y(t) = 0 \qquad \text{and} \qquad E_D(t) = \int_0^t Y(t')\, dt' \ = \ 0 \tag{151),(152}$$

for all $t \geq 0$. Inserting equation (151) and (152) into (141), we have

$$E_D(t) = \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{D}(\mathbf{x}, t) \cdot \mathbf{D}(\mathbf{x}, t)\ d^3\mathbf{x} = 0 \tag{153}$$

Since the integrand $\mathbf{D}(\mathbf{x}, t) \cdot \mathbf{D}(\mathbf{x}, t)$ in this equation is greater than or equal to zero and is continuous in $\mathbf{x}$ over all $\mathbb{R}^3$ we must have $\mathbf{D}(\mathbf{x}, t) \cdot \mathbf{D}(\mathbf{x}, t) = 0$ for all $\mathbf{x}$ and t. Inserting this result into equation (135), we then have $\mathbf{u}^{(1)}(\mathbf{x}, t) = \mathbf{u}^{(2)}(\mathbf{x}, t)$ for all $\mathbf{x}$ and t, and therefore the solution is unique. Since this difference $\mathbf{D}(\mathbf{x}, t)$ between the solutions $\mathbf{u}^{(1)}(\mathbf{x}, t)$ and $\mathbf{u}^{(2)}(\mathbf{x}, t)$ is identically zero, if follows that the solution $\mathbf{u}(\mathbf{x}, t)$ is unique. Hence, the uniqueness part of our main theorem is proven, which concludes our proof of this theorem.

Conclusion

In this paper, we have shown existence of a solution of the zero driving-force Navier-Stokes equation in free space with given initial fluid velocity and spatial derivatives profiles which approach zero as $a^\kappa/(|\mathbf{x}| + a)^\kappa$ as $|\mathbf{x}| \to \infty$, assuming a scalar pressure and incompressibility of the fluid. Existence of a smooth, finite energy solution was proven by first establishing that such a solution would retain this spatial characteristic when propagated over any finite time interval. Next, it was proven that the solution $\mathbf{u}(\mathbf{x}, t)$ must be bounded by showing that the time integral of the scalar pressure gradient ∇p remains bounded and continuous despite possible irregularities in the solution components u_i and their spatial derivatives. Finally, we showed that the solution is unique.

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